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APPLICATION OF THE NEW EDUCATION TO THE DIFFERENTIAL AND INTEGRAL CALCULUS.

By **FLETCHER DURELL, Ph. D., Professor of Mathematics, Dickinson College, Carlisle, Pennsylvania.**

[Read before the New York Mathematical Society, November 4, 1893.]

At a recent meeting* of the Physical Society of Great Britain, a paper was read by Mr. F. W. Sanderson, in which was presented a method of teaching mathematics in preparation for science teaching. In the paper and in the discussion which followed, the methods of the so-called new education were emphatically endorsed. The student in each new advance is to begin with the concrete object, something which he can see and handle and perhaps make, and go on to abstractions only for the sake of realized advantages. Drawing is to precede formal Geometry (Euclid) and plotting of curves is to precede Analytical Geometry. In the course of the discussion it was remarked that the "plotting of curves might be made a means of furnishing the fundamental notions in the Differential and Integral Calculus". As a matter of fact, the writer has been using a method of this sort for some years in the class room.

It being supposed that the pupil has been made thoroughly familiar with methods of drawing graphs of all ordinary algebraic and trigonometric functions, ideas of continuity and discontinuity having been thus formed to some extent by induction, the subject of the Differential and Integral Calculus may then be approached as follows: Take some simple convenient curve as $y=x^2+1$ ....(1) and trace it. Let the student recall the method of finding the slope of the tangent of this curve, given in most American text-books on Analytical Geometry; that is, take two points on the curve, P or (x_1,y_1), and Q or (x_2,y_2) and obtain the slope of the secant through them, $\frac{y_2-y_1}{x_2-x_1}=x_2+x_1$. Let the point Q move along the curve till it coincides† with P, then must the slope of the tangent at $P=2x_1$. But this method of finding the slope of a curve fails, or, at least, becomes very difficult for many curves as, for example, $y=\sin x^2$. Hence, in order to obtain greater power, we modify the method thus: denote P by (x,y), Q by $(x+\Delta x, y+\Delta y)$, then by a precisely similar use of the equation obtained, we get $\frac{\Delta y}{\Delta x}=2x+\Delta x$...(2), and the slope of the tangent at $P=2x$. The difference between the two methods and the advantage of the second method is seen to lie in this: In the first method the co-ordinates of the two points, P and Q, viz.; x_1, x_2, y_1, y_2, are made primary in the notation, and we are left to work out the differences of these which determine the slope of the secant as best we may. But in the second method, these differences, Δx and Δy are made primary in the notation, and appear in the first equations; hence, the whole work can readily be guided with reference to these differences and the determination of their ratio.

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* January 13, 1893. See NATURE, Vol. 47, pp. 358–9.

† See the discussion of the propriety of making the two points actually coincide, given later, pp. 16–17.

In this advantage lies the germ of the Differential Calculus. In order to fully grasp and realize this, let the student find the slopes, by this differential method, of curves like $y=x^3$ and $y=\frac{x+1}{x}$, trace them and draw tangents at indicated points, by use of the slope values obtained.

Differentiation may now be provisionally defined as the method of finding the slope of a curve by the differential method.

We next proceed to apply the differential method to finding the slopes of curves in general groups. In order to do so we suppose general relations to exist among curves thus: in Fig. 1, if each ordinate of (c) equals the sum of the corresponding ordinates of (a) and (b), as $PM=AM+GM$, $QN=BN+DN$, etc., (c) may be regarded as the sum of (a) and (b) by addition of ordinates. If, instead, the number of units in each ordinate of (c) =the product of the numbers of units in the corresponding ordinates of (a) and (b), (c) may be regarded in like manner as the product of (a) and (b). Thus all the other elementary functional relations as quotient, power, root, etc., may be conceived of as existing among curves.

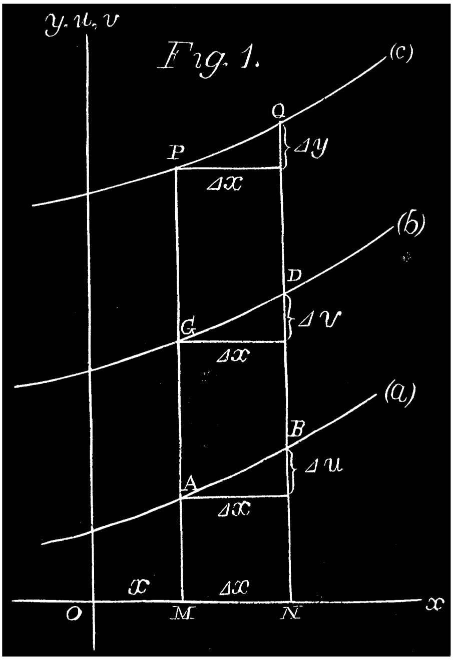
Fig. 1.

If now the equation of (a) be $u=f(x)$,

of (b), $v=F(x)$,

of (c), $y=u+v$,

we obtain as usual, $\frac{\Delta y}{\Delta x}=\frac{\Delta u}{\Delta x}+\frac{\Delta v}{\Delta x}$, that is, from Fig. 1, slope sec. PQ=sl. sec. GD+sl. sec. AB. Let Q move to P, then *must* sl. tan. at P=sl. tan. at G+sl. tan. at B, or, using symbols, $\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$ ....(3). Hence, if the slopes of (a) and (b) are known, that of (c) can be at once written down.

Let us now pause* to notice the matter of letting the secant become a tangent and why when $\Delta y, \Delta x$ and the other differences become zero, we can still determine the slopes of tangents at P, A, and G. Does the secant actually become a tangent or does it only approach the tangent as a limiting position? Some writers say that it actually becomes a tangent; others that it only approaches it as a limit; still others attempt to make compromises, but with results which are puzzling and contradictory.* But it is plain that a secant can become a tangent and that theorems can be directly demonstrated thereby. To give a simple illustration we will prove, by the use of this principle, that the radius of a circle is perpendicular to the tangent at its extremity. Draw the secant PQ of a given circle, OP and OQ radii, and OR the perpendicular from the center O on PQ. Since OPQ is an isosceles triangle, OR bisects the base PQ, $\therefore$ R lies between P and Q. Let Q move along the circumference to P; when Q coincides with P, R also coincides with P; that is, OR the perpendicular on the secant becomes OP, the perpendicular on the tangent, and at the same time becomes the radius to the point of contact; hence, the radius of the point of contact is perpendicular to the tangent.

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*This digression is not necessary in teaching a class if the matter has been properly presented in teaching Analytical Geometry.

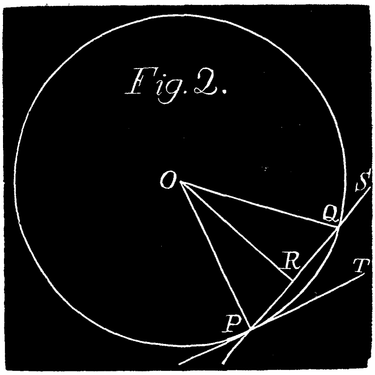
Fig. 2.

Similarly, if the secant of a curve through the points (x_1,y_1) and (x_2,y_2) have an intercept on the axis of x equal to $\frac{3}{x_1+x_2}$, the intercept on the same axis of the tangent at (x_1,y_1) is $\frac{3}{2x_1}$.

Thus a property of a tangent may be obtained directly from the property of the secant, by transforming the secant into a tangent.

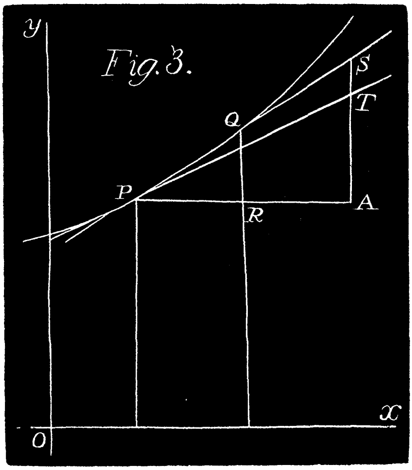
Fig. 3.

But in some cases a modification of this method needs to be introduced. In the Calculus, we need to use negative or indirect reasoning also. Greater generality of treatment and greater facility of transformation and reduction are obtained by determining the direction of the secant by the use of the quantities QR and PR (Fig. 3) than in any other way; the slope of the secant is then represented by $\frac{QR}{PR}$ which appears in symbols as $\frac{\Delta y}{\Delta x}$, $\frac{\Delta u}{\Delta x}$, $\frac{\Delta v}{\Delta x}$ etc. These ratios become $\frac{0}{0}$ when Q is made to coincide with P. If $PA=1$, SA represents the slope of the secant PQ, and TA that of the tangent at P. When the secant becomes the tangent at P, SA becomes TA directly; but inasmuch as the relation between SA and TA (and similarly in similar cases) is proved by means of QR and PR which vanish, our method of proving the value of SA must be negative or indirect. The equation which contains PR and QR as finite quantities shows that the value of TA can be neither greater nor less than what the value of the slope of the secant becomes and therefore equals it. Thus in the case of $y=x^2+1$, given above, the slope of the tangent at (x,y) must $=2x$, for if it $=2x \pm h$, the so-called tangent, by equation (2), would be the secant through the points whose abscissas are x and $x+h$, or x and $x-h$,* and hence would not be the tangent at all. It is in the sense of this negative reasoning that the word "must" is used in the above demonstration. Observe that in the theorem proved from Fig. 2, the auxiliary quantities OP and OQ do not vanish, hence, direct reasoning is used throughout; but in proving a theorem from the data of Fig. 3, the auxiliary quantities PR and QR vanish, and hence negative reasoning is employed.

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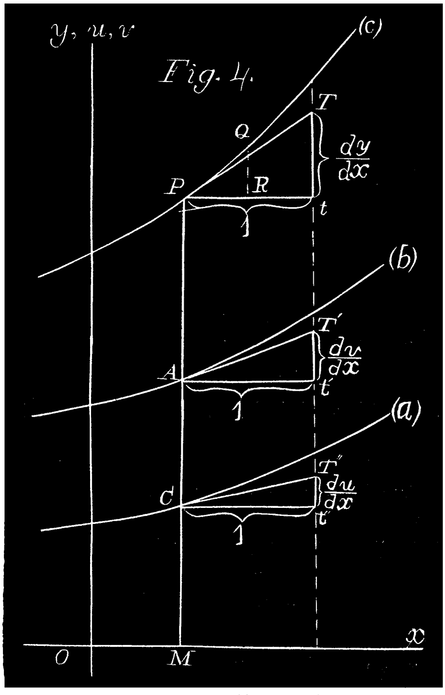
* Thus in the Differential Calculus, by J. Edwards, p. 7, we find the following defini-

The advantage of this over the ordinary presentation by the method of limits is that the latter is too often a mere approximation in the mind of the student, while in the method here presented, there is, so to speak, absolute *contact* between the limit and the determining varaible, as well as continuity between other parallel variables and the same limit.

The result obtained in (3) may now be expressed thus, that the student may more firmly grasp the relation of the quantities involved: $Tt=T't'+T''t''$, $\Delta y, \Delta x$, etc., are seen to be merely the auxiliary lines PR, QR, etc., by which the result is established. The student should verify this result by several constructions of curves and their tangents.



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tion of a tangent: "Let PQ be a chord joining P and Q, two adjacent points on a curve. Let Q travel along the curve towards P, and come so close as *ultimately to coincide with P.* Then the *limiting position of PQ*, viz.; PT is called the tangent at P."

Salmon says, Conic Sections. Sixth Ed., p. 78, "we shall not say that the tangent meets the circle in only one point, but rather that it meets it in *two coincident* points....And in general we define the tangent to any curve as the line joining *two indefinitely near points* on the curve."

It will appear in the subsequent investigation that the definition which we provisionally make of a tangent is contained in the first sentence of Mr. Edwards' definition, omitting the word "ultimately." The use of this word, derived though it be from Newton, seems like trying to get rid of a difficulty by postponing it indefinitely in time.

* At a cusp point, this reasoning is somewhat modified. At this point the tangent on being rotated backward must at once become a secant, cutting the curve in an initially adjacent point; hence the slope of the tangent must be such that on being in the slightest degree increased or decreased, in one of these cases it becomes the slope of a secant.

If now, using Fig. 1 in a new way, (*c*) be taken as the product of (*a*) and (*b*) by multiplication of ordinates, we readily obtain, by a similar method of proof, the result,

$$\frac{dy}{dx}=u\frac{dv}{dx}+v\frac{du}{dx}.$$

This result is expressed to the eye by a figure similar to Fig. 4, wherein the number of linear spaces in Tt=(the number of linear spaces in CM)×(the number in $T't'$)+(number in $T''t''$)×(the number in AM).

Thus when the slopes of two curves are known, that of their product can be at once written down.

In this way all the ordinary formulas of differentiation are obtained as applied to curves.

The student should now be introduced to the idea of function, that is of quantity independent of any spacial relations, and brought to see the greater freedom and flexibility of treatment obtained by its use.

(TO BE CONTINUED.)

ARITHMETIC.

Conducted by B. F. FINKEL, Kidder, Missouri. All contributions to this department should be sent to him.

SOLUTIONS TO PROBLEMS.

1. I. Reduce $\frac{1}{3}$ to 8ths.

1st Solution.	2nd Solution.
II. 1. $1=8$ eighths.	II. 1. $1=\frac{8}{8}$.
II. 2. $\frac{1}{3}=\frac{1}{3}$ of 8 eighths$=2\frac{2}{3}$ eighths.	II. 2. $\frac{1}{3}=\frac{1}{3}\times\frac{8}{8}=\frac{2\frac{2}{3}}{8}.$
III. ∴ $\frac{1}{3}=2\frac{2}{3}$ eighths, or $\frac{2\frac{2}{3}}{8}$.	III. ∴ $\frac{1}{3}=\frac{2\frac{2}{3}}{8}$.

2. I. Reduce $\frac{5}{6}$ to a fraction whose numerator is 11.

II. 1. $1=\frac{11}{11}$

II. 2. $\frac{5}{6}=\frac{5}{6}\times\frac{11}{11}=\frac{11}{\frac{55}{6}}=\frac{11}{9\frac{1}{6}}.$

III. ∴ $\frac{5}{6}=\frac{11}{9\frac{1}{6}}$, a fraction whose numerator is 11.

3. **Proposed by L. B. Hayward, Superintendent of Schools, Bingham, Ohio.**

Bought an article and sold it for 3% less than it cost me; bought it back paying 3% more than I sold it for. I lost $12.00 by the transaction. What did the article cost at first?

Solution by the Proposer.

Let 100% = the cost at first.
Then 97% = selling price.